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Kac–Moody–Virasoro symmetry algebra and symmetry reductions of the bilinear sinh-Gordon equation in (2 + 1)-dimensions

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Abstract

By means of the formal series symmetry approach proposed in [1], infinite many symmetries and the corresponding Kac–Moody–Virasoro Lie symmetry algebra of a new bilinear (2 + 1)-dimensional sinh-Gordon equation are given. Then, the obtained symmetries are used to get the symmetry reductions of the model. From one of the special reduction we know that the bilinear form of the first member of the negative Kadomtsev–Petviashvili hierarchy is not only a (2 + 1)-dimensional sinh-Gordon extension but also a novel (2 + 1)-dimensional classical Boussinesq extension.

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1. Introduction

The so-called integrable systems [2] such as the Korteweg-de Vries (KdV) equation, the Nizhnik–Novikov–Veselov (NNV) equation, the Kadomtsev–Petviashvili (KP) equation, the nonlinear Schrödinger (NLS) equation, the sine-Gordon (sG) equation and the sinh-Gordon (ShG) equation have been found in various application fields including the relativistic field theory, the string dynamics, the hydrodynamics, the thermodynamics, the solid-state physics, the nonlinear optics and so on [3–11].

It is well known that the (1 + 1)-dimensional KdV equation possesses several (2 + 1)-dimensional integrable extensions, say, the KP equation, the NNV system, the asymmetric NNV equation, the breaking soliton equation, etc. Furthermore, it is also known that the (1 + 1)-dimensional sG and/or ShG equation are equivalent to the first one of the negative KdV hierarchy via the Miura transformation. Thus, it is natural that every (2 + 1)-dimensional integrable extension of the (1 + 1)-dimensional KdV system will lead to an integrable (2 + 1)-dimensional sG or ShG equation from its negative hierarchy. Doing more research into sG and ShG equation systems is no doubt a quite interesting and important assignment, especially in high dimensions, on which few researchers have done much work.

In fact, some (2 + 1)-dimensional sG and/or ShG models have been found, for example, the authors of [15] have proposed a (2 + 1)-dimensional sG equation (which is equivalent to one of the negative NNV hierarchy) in a rather symmetric manner and the author of [16] gives us another one (which corresponds to one of the negative breaking soliton hierarchy) but without spatial symmetry of x and y. Here, what interests us much more is the (2 + 1)-dimensional ShG equation (which is related to the first one of the negative NKP hierarchy) presented in [17]:

$$[\alpha\phi_{yt} + \mathrm{e}^{-2\phi}(\mathrm{e}^{2\phi}\phi_{xt})_x]_y = -(s_x\,\mathrm{e}^{2\phi})_{xx}, \tag{1}$$

$$\left[\mathrm{e}^{2\phi}\left(\phi_{xt} - \tfrac{1}{2}C_1\,\mathrm{e}^{2\phi} + \tfrac{1}{2}C_2\,\mathrm{e}^{-2\phi}\right)\right]_x + \alpha\,\mathrm{e}^{2\phi}\phi_{yt} + \alpha\,\mathrm{e}^{2\phi}(\mathrm{e}^{2\phi}s)_x = 0, \tag{2}$$

where C_1, C_2 and α are arbitrary constants.

Without loss of the generality, we can choose

$$2\phi = \omega, \qquad C_1 = C_2 = \tfrac{1}{2}, \qquad \alpha = 1, \qquad s = \theta,$$

then the model equation becomes

$$(\omega_{xt} - \sinh\omega)_x + \omega_x(\omega_{xt} - \sinh\omega) + \omega_{yt} = -2(\theta\,\mathrm{e}^{\omega})_x, \tag{3}$$

$$2[(\theta\,\mathrm{e}^{\omega})_x - \theta\,\mathrm{e}^{\omega}\omega_x]_{xx} + \left(\omega_y + \omega_{xx} + \tfrac{1}{2}\omega_x^2\right)_{yt} = 0. \tag{4}$$

For this ShG equation, the authors of [18] have presented us its bilinear form

$$\left(D_y + D_x^2\right)F\cdot G = 0, \tag{5}$$

$$D_t\left(D_y + D_x^2\right)F\cdot G + 2D_xF\cdot G = 0, \tag{6}$$

in which $F \equiv F(x, y, t)$ and $G \equiv G(x, y, t)$, and we default them in the later part of the paper for convenience. F, G, ω and θ are related by

$$\omega(x, y, t) = 2\ln\frac{F(x, y, t)}{G(x, y, t)} \tag{7}$$

$$\theta(x, y, t) = 2\int_{-\infty}^{x}[\ln G(\xi, y, t)]_{yt}\left[\frac{G(\xi, y, t)}{F(\xi, y, t)}\right]^2\mathrm{d}\xi \tag{8}$$

and the Hirota's bilinear operators D_x, D_y, D_t are defined as

$$D_x^mD_y^nD_t^kF\cdot G = \partial_{\epsilon_1}^m\partial_{\epsilon_2}^n\partial_{\epsilon_3}^kF(x+\epsilon_1, y+\epsilon_2, t+\epsilon_3)\cdot G(x-\epsilon_1, y-\epsilon_2, t-\epsilon_3)|_{\epsilon_1=0,\epsilon_2=0,\epsilon_3=0}.$$

Considering further, if we add an arbitrary function $q(y, t)$ to equation (8),

$$\theta(x, y, t) = 2\int_{-\infty}^{x}[\ln G(\xi, y, t)]_{yt}\left[\frac{G(\xi, y, t)}{F(\xi, y, t)}\right]^2\mathrm{d}\xi + q(y, t), \tag{9}$$

it is not difficult to find that the changed $\theta(x, y, t)$ (9) with the unchanged $\omega(x, y, t)$ (7) can still bilinearize the (2 + 1)-dimensional ShG equation system (3) and (4).

In this paper, we mainly resolve the generalized symmetries of the new (2 + 1)-dimensional bilinear form by means of the formal series approach [1], not the mastersymmetry approach [12] nor the recursion operator method [13] for both of which are quite difficult to apply in (2 + 1)-dimensional cases.

The general structure of the paper is organized as follows: in section 2, at first, we give the new bilinear form of the (2 + 1)-dimensional equation system (3) and (4) under the new transformation relation, equation (9), and the old one, equation (7). Then, we calculate the corresponding generalized symmetries of the bilinear (2 + 1)-dimensional ShG model and its Kac–Moody–Virasoro Lie algebra. In section 3, we present the symmetry reduction results. Finally, in section 4, we make a short summary and propose some questions that need to be solved in the future.

2. Bilinear form and corresponding generalized symmetries

Here, we just give a more universal bilinear form of the (2 + 1)-dimensional ShG system (3) and (4) without concrete proof:

$$\left(D_y + D_x^2\right)F \cdot G = 0, \tag{10}$$

$$D_t\left(D_y + D_x^2\right)F \cdot G + qD_xF \cdot G = 0, \tag{11}$$

where $q \equiv q(y,t)$, and $F, G, \omega(x,y,t)$ and $\theta(x,y,t)$ are related by equations (7) and (9), i.e.,

$$\omega(x,y,t) = 2\ln\frac{F(x,y,t)}{G(x,y,t)},$$

$$\theta(x,y,t) = 2\int_{-\infty}^{x}[\ln G(\xi,y,t)]_{yt}\left[\frac{G(\xi,y,t)}{F(\xi,y,t)}\right]^2 \mathrm{d}\xi + q(y,t).$$

Now, we concentrate on looking for the symmetries of the bilinear (2 + 1)-dimensional ShG equation by formal series approach.

A symmetry

$$\sigma = \begin{pmatrix}\sigma_F\\ \sigma_G\end{pmatrix}$$

of the bilinear KP equation is defined as a solution of its linearized form

$$\left(D_y + D_x^2\right)F \cdot \sigma_G - \left(D_y - D_x^2\right)G \cdot \sigma_F = 0, \tag{12}$$

$$D_t\left(D_x^2 + D_y\right)F \cdot \sigma_G - D_t\left(D_x^2 - D_y\right)G \cdot \sigma_F + qD_x(F \cdot \sigma_G - G \cdot \sigma_F) = 0. \tag{13}$$

According to the formal series symmetry expansion approach, σ_F and σ_G can be expanded as

$$\sigma_F = \sum_{k=0}^{\infty} f^{(n-k)}\sigma_{1n}[k], \tag{14}$$

$$\sigma_G = \sum_{k=0}^{\infty} f^{(n-k)}\sigma_{2n}[k], \tag{15}$$

where f is an arbitrary function of the spatial variable y and $f^{(n-k)}$ denotes $(n-k)$th derivative of f, while σ_{1n} and σ_{2n} are functions of x, t, F, G and arbitrary derivatives of F and G but not y dependent explicitly. Clearly, the y part has been separated out to $f^{(n-k)}$.

Remark. We may change the function f in the series expansion of σ_G as a different arbitrary function $g(y)$. Nevertheless, the detailed calculations show us that there is no such arbitrary functions in the expansions.

Substituting the above equations (14) and (15) into symmetry definition equations (12) and (13), we obtain

$$\sum_{k=1}^{\infty} f^{n-k+1}\Big\{\big(D_y + D_x^2\big)F\cdot\sigma_{2n}[k-1] - \big(D_y - D_x^2\big)G\cdot\sigma_{1n}[k-1]$$

$$-\sum_{k=0}^{\infty} f^{n-k+1}(F\sigma_{2n}[k] - G\sigma_{1n}[k])\Big\} = 0, \tag{16}$$

$$\sum_{k=1}^{\infty} f^{n-k+1}\Big\{\big[D_t\big(D_y + D_x^2\big) + qD_x\big]F\cdot\sigma_{2n}[k-1] + \big[D_t\big(D_y - D_x^2\big) - qD_x\big]G\cdot\sigma_{1n}[k-1]$$

$$-\sum_{k=0}^{\infty} f^{(n-k+1)} D_t(F\cdot\sigma_{2n}[k] + G\cdot\sigma_{1n}[k])\Big\} = 0. \tag{17}$$

Because of the arbitrariness of the function f, all coefficients of the same arbitrary order derivatives of f in equations (16) and (17) must vanish, i.e.,

$$-F\sigma_{2n}[0] + G\sigma_{1n}[0] = 0, \tag{18}$$

$$-D_t(F\sigma_{2n}[0] + G\sigma_{1n}[0]) = 0, \tag{19}$$

$$\big(D_y + D_x^2\big)F\cdot\sigma_{2n}[k-1] - \big(D_y - D_x^2\big)G\cdot\sigma_{1n}[k-1] - F\sigma_{2n}[k] + G\sigma_{1n}[k] = 0, \tag{20}$$

$$\begin{aligned} &D_t\big(D_y + D_x^2\big)F\cdot\sigma_{2n}[k-1] + D_t\big(D_y - D_x^2\big)G\cdot\sigma_{1n}[k-1] + qD_x(F\cdot\sigma_{2n}[k-1] \\ &\quad - G\cdot\sigma_{1n}[k-1]) - D_t(F\cdot\sigma_{2n}[k] + G\cdot\sigma_{1n}[k]) = 0 \quad (k = 1, 2, \ldots). \end{aligned} \tag{21}$$

Solving the above equations (18)–(21) we can find that

$$\sigma_{1n}[0] = Fg_{n0}(x, y), \qquad \sigma_{2n}[0] = Gg_{n0}(x, y),$$

$$\sigma_{1n}[k] = -\frac{a}{G} + F\int\left(\frac{-G_t a}{FG^2} + \frac{a_t - b}{2FG}\right)\mathrm{d}t,$$

$$\sigma_{2n}[k] = G\int\left(\frac{-G_t a}{FG^2} + \frac{a_t - b}{2FG}\right)\mathrm{d}t,$$

where

$$a = \big(D_y + D_x^2\big)F\cdot\sigma_{2n}[k-1] - \big(D_y - D_x^2\big)G\cdot\sigma_{1n}[k-1]$$

and

$$b = \big\{D_t\big(D_y + D_x^2\big) + qD_x\big\}F\cdot\sigma_{2n}[k-1] + \big\{D_t\big(D_y - D_x^2\big) - qD_x\big\}G\cdot\sigma_{1n}[k-1].$$

To see the recursion relation clearly, we write above recursion equations in another equivalent form

$$\begin{pmatrix}\sigma_{1n}[0]\\ \sigma_{2n}[0]\end{pmatrix} = \begin{pmatrix}Fg_{n0}(x, y)\\ Gg_{n0}(x, y)\end{pmatrix}, \tag{22}$$

$$\begin{pmatrix}\sigma_{1n}[k]\\ \sigma_{2n}[k]\end{pmatrix} = \begin{pmatrix}-\frac{1}{G} + F\partial_t^{-1}\big(-\frac{G_t}{FG^2} + \frac{\partial_t}{2FG}\big) & -F\partial_t^{-1}\frac{1}{2FG}\\ -G\partial_t^{-1}\big(-\frac{G_t}{FG^2} + \frac{\partial_t}{2FG}\big) & -G\partial_t^{-1}\frac{1}{2FG}\end{pmatrix}^k \cdot \begin{pmatrix}a\\ b\end{pmatrix}, \tag{23}$$

$$\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\big(D_y + D_x^2\big)F & -\big(D_y + D_x^2\big)G\\ D_t\big(D_y + D_x^2\big)F + qD_xF & D_t\big(D_y - D_x^2\big)G - qD_xG\end{pmatrix}^k \cdot \begin{pmatrix}\sigma_{1n}[0]\\ \sigma_{2n}[0]\end{pmatrix}. \tag{24}$$

And then the series symmetry has the form

$$\begin{pmatrix}\sigma_{nF}\\ \sigma_{nG}\end{pmatrix} = \sum_{k=0}^{\infty} f^{(n-k)} \cdot \begin{pmatrix} -\frac{1}{G} + F\partial_t^{-1}\left(-\frac{G_t}{FG^2} + \frac{\partial_t}{2FG}\right) & -F\partial_t^{-1}\frac{1}{2FG} \\ -G\partial_t^{-1}\left(-\frac{G_t}{FG^2} + \frac{\partial_t}{2FG}\right) & -G\partial_t^{-1}\frac{1}{2FG}\end{pmatrix}^k$$
$$\cdot \begin{pmatrix} \left(D_y + D_x^2\right)F & -\left(D_y + D_x^2\right)G \\ D_t\left(D_y + D_x^2\right)F + qD_xF & D_t\left(D_y - D_x^2\right)G - qD_xG\end{pmatrix}^k \cdot \begin{pmatrix}\sigma_{1n}[0]\\ \sigma_{2n}[0]\end{pmatrix}. \tag{25}$$

Now, we look for the concerned truncated symmetries. In the course of calculating, if there exists one set of $\sigma_{1n}[j]$ and $\sigma_{2n}[j]$ which itself is a symmetry, i.e.,

$$\left(D_y + D_x^2\right)F \cdot \sigma_{2n}[j] - \left(D_y - D_x^2\right)G \cdot \sigma_{1n}[j] = 0 \tag{26}$$

and

$$D_t\left(D_x^2 + D_y\right)F \cdot \sigma_{2n}[j] - D_t\left(D_y - D_x^2\right)G \cdot \sigma_{1n}[j] + qD_x(F \cdot \sigma_{2n}[j] - G \cdot \sigma_{1n}[j]) = 0, \tag{27}$$

we say that the symmetry is truncated because all $\sigma_{1n}[k]$ and $\sigma_{2n}[k]$ can be taken as zero for arbitrary $k > j$.

Fortunately, we find that when we take

$$g_n(x, y) = \frac{x^n}{n!}, \qquad n = 0, 1, 2, 3, \ldots \tag{28}$$

and

$$q(y, t) = q(t), \tag{29}$$

the series symmetries given by (25) are truncated one.

Thus now, with the help of the recursion equation (25) including equations (22), (29) and (30), we can obtain infinitely many truncated symmetries of the bilinear (2 + 1)-dimensional ShG equations (10) and (11). For a clear understanding, we write the first six truncated symmetries,

$$\sigma_0(f) = \begin{pmatrix} fF \\ fG \end{pmatrix}, \tag{30}$$

$$\sigma_1(f) = \begin{pmatrix} fxF \\ fxG \end{pmatrix}, \tag{31}$$

$$\sigma_2(f) = \begin{pmatrix} \frac{1}{2}f_yx^2F - fF \\ \frac{1}{2}f_yx^2G + fG \end{pmatrix}, \tag{32}$$

$$\sigma_3(f) = \begin{pmatrix} \frac{1}{6}f_{yy}x^3F - f_yxF + f\left(F\int q\,\mathrm{d}t + 4F_x\right) \\ \frac{1}{6}f_{yy}x^3G + f_yxG + f\left(G\int q\,\mathrm{d}t + 4G_x\right) \end{pmatrix}, \tag{33}$$

$$\sigma_4(f) = \begin{pmatrix} \frac{1}{24}f_{yyy}x^4F - \frac{1}{2}f_{yy}x^2F + f_yx\left(F\int q\,\mathrm{d}t + 4F_x\right) + 8fF_y \\ \frac{1}{24}f_{yyy}x^4G + \frac{1}{2}f_{yy}x^2G + f_yx\left(G\int q\,\mathrm{d}t + 4G_x\right) + 8fG_y \end{pmatrix}, \tag{34}$$

$$\sigma_5(f) = \begin{pmatrix} \sigma_{15}(f) \\ \sigma_{25}(f) \end{pmatrix}, \tag{35}$$

where $\sigma_{15}(f)$ and $\sigma_{25}(f)$ are as follows:

$$\begin{aligned}
\sigma_{15}(f) = & -\frac{1}{120} f_{yyyy} x^5 F - f_{yy} x^2 \left(\frac{1}{2} F \int q \, \mathrm{d}t + 2F_x\right) + f_y \left(F \int q \, \mathrm{d}t + 4F_x - 8x F_y\right) \\
& + \frac{1}{6} f_{yyy} x^3 F + 4f \left[F \left(-4 \int \frac{F_x F_{yt}}{F^2} \mathrm{d}t + 4 \int \frac{F_{xt} F_{xx}}{F^2} \mathrm{d}t - 8 \int \frac{F_x G_y G_t}{FG^2} \mathrm{d}t \right.\right. \\
& + 4 \int \frac{G_x F_y F_t}{F^2 G} \mathrm{d}t + \int \frac{q G_x^2}{G^2} \mathrm{d}t + 4 \int \frac{F_x F_y F_t}{F^3} \mathrm{d}t - 4 \int \frac{G_x F_{yt}}{FG} \mathrm{d}t \\
& - 8 \int \frac{F_x F_{xt} G_x}{F^2 G} \mathrm{d}t - 2 \int \frac{q G_y}{G} \mathrm{d}t - 2 \int \frac{q G_x F_x}{FG} \mathrm{d}t + 8 \int \frac{F_x^2 F_t G_x}{F^3 G} \mathrm{d}t \\
& + 8 \int \frac{F_x G_{yt}}{FG} \mathrm{d}t - 4 \int \frac{F_x F_t F_{xx}}{F^3} \mathrm{d}t + \int \frac{q F_{xx}}{F} \mathrm{d}t - 8 \int \frac{F_x G_x G_{xt}}{FG^2} \mathrm{d}t \\
& \left.\left. + 8 \int \frac{F_x G_x^2 G_t}{FG^3} \mathrm{d}t + 2 \int \frac{q F_y}{F} \mathrm{d}t \right) + 4 \frac{F_x F_y}{F} + 4 \frac{F_x G_x^2}{G^2} - 8 \frac{F_x G_y}{G} + 4F_{xy} \right]
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\sigma_{25}(f) = & -\frac{1}{120} f_{yyyy} x^5 G - \frac{1}{6} f_{yyy} x^3 G - f_{yy} x^2 \left(\frac{1}{2} G \int q \, \mathrm{d}t + 2G_x\right) \\
& - f_y \left(G \int q \, \mathrm{d}t + 4G_x + 8x G_y\right) + f \left[G \left(-4 \int \frac{F_x F_{yt}}{F^2} \mathrm{d}t + 4 \int \frac{F_{xt} F_{xx}}{F^2} \mathrm{d}t \right.\right. \\
& - 8 \int \frac{F_x G_y G_t}{FG^2} \mathrm{d}t + 4 \int \frac{G_x F_y F_t}{F^2 G} \mathrm{d}t + 4 \int \frac{F_x F_y F_t}{F^3} \mathrm{d}t - 4 \int \frac{G_x F_{yt}}{FG} \mathrm{d}t \\
& - 8 \int \frac{F_x F_{xt} G_x}{F^2 G} \mathrm{d}t + 8 \int \frac{F_x^2 F_t G_x}{F^3 G} \mathrm{d}t + 8 \int \frac{F_x G_{yt}}{FG} \mathrm{d}t - 4 \int \frac{F_x F_t F_{xx}}{F^3} \mathrm{d}t \\
& - 8 \int \frac{F_x G_x G_{xt}}{FG^2} \mathrm{d}t + 8 \int \frac{F_x G_x^2 G_t}{FG^3} \mathrm{d}t + \int \frac{q G_x^2}{G^2} \mathrm{d}t - 2 \int \frac{q G_y}{G} \mathrm{d}t \\
& \left. - 2 \int \frac{q G_x F_x}{FG} \mathrm{d}t + \int \frac{q F_{xx}}{F} \mathrm{d}t + 2 \int \frac{q F_y}{F} \mathrm{d}t \right) - 2 \frac{G F_x F_{xx}}{F^2} + 2 \frac{G F_x F_y}{F^2} \\
& + 4 \frac{F_x G_x^2}{FG} - 4 \frac{F_x G_y}{F} + 2 \frac{G F_{xy}}{F} + 4 \frac{F_y G_x}{F} + 4 \frac{F_x^2 G_x}{F^2} + 2 \frac{G F_{xxx}}{F} \\
& \left. - 4G_{xy} - 4 \frac{F_{xx} G_x}{F} \right].
\end{aligned} \tag{37}$$

In calculating, we find that to get higher truncated symmetries (large n) from the recursion equation (25) is formidable because of the difficulty in determining the integration function, though it is feasible in principle.

From the above six truncated symmetries, it is not difficult to prove that $\sigma_0 - \sigma_4$ constitute an infinite-dimensional closed Kac–Moody–Virasoro-type Lie symmetry algebra [14]:

$$\begin{aligned}
[\sigma_0(f_1), \sigma_1(f_2)] &= [\sigma_0(f_1), \sigma_2(f_2)] = [\sigma_0(f_1), \sigma_3(f_2)] = [\sigma_1(f_1), \sigma_2(f_2)] \\
&= [\sigma_0(f_1), \sigma_0(f_2)] = [\sigma_1(f_1), \sigma_1(f_2)] = [\sigma_2(f_1), \sigma_2(f_2)] = 0, \\
[\sigma_4(f_1), \sigma_4(f_2)] &= 8\sigma_4(f_{1y} f_2 - f_1 f_{2y}), \\
[\sigma_3(f_1), \sigma_3(f_2)] &= 4\sigma_2(f_{1y} f_2 - f_1 f_{2y}), \\
[\sigma_0(f_1), \sigma_4(f_2)] &= -8\sigma_0(f_2 f_{1y}), \\
[\sigma_1(f_1), \sigma_3(f_2)] &= -4\sigma_0(f_1 f_2),
\end{aligned}$$

$$[\sigma_1(f_1), \sigma_4(f_2)] = -4\sigma_1(f_1 f_{2y} + 2f_{1y} f_2),$$
$$[\sigma_2(f_1), \sigma_3(f_2)] = -4\sigma_1(f_{1y} f_2),$$
$$[\sigma_2(f_1), \sigma_4(f_2)] = 8\sigma_2(f_{1y} f_2),$$
$$[\sigma_3(f_1), \sigma_4(f_2)] = \tfrac{1}{4}\sigma_2(-2f_{1y} f_2 + f_{2y} f_1),$$

where the commutator is defined as

$$[A(u), B(u)] = \frac{\partial}{\partial \epsilon}[A(u + \epsilon B(u)) - B(u + \epsilon A(u))]\bigg|_{\epsilon=0}.$$

According to the general theory [19], if $A(u)$ and $B(u)$ are symmetries, so is the commutator $[A(u), B(u)]$, from which we can obtain infinitely many generalized symmetries of the bilinear (2 + 1)-dimensional ShG model equations (10) and (11) in the following way:

$$\sigma_{n+1}(f) = [\sigma_5(f_1), \sigma_n(1)], \qquad f = \dot{f}_1. \tag{38}$$

As the case in calculating higher order symmetries σ_{1n} and σ_{2n} (large n) from the recursion equation (25), it is also rather arduous to obtain higher order symmetries from the commutation relation (38) for the same reason of the difficulty in determining the integration function. In fact, the truncated symmetries deduced from the commutation relation (38) are the same as those obtained from the recursion relation (25).

Remark. We can also do formal series expansions with arbitrary functions of x or t, namely,

$$\sigma_F = \sum_{k=0}^{\infty} P(x)^{(n-k)}\sigma_{1n}[k], \qquad \sigma_G = \sum_{k=0}^{\infty} P(x)^{(n-k)}\sigma_{2n}[k] \tag{$\star$}$$

or

$$\sigma_F = \sum_{k=0}^{\infty} Q(t)^{(n-k)}\sigma_{1n}[k], \qquad \sigma_G = \sum_{k=0}^{\infty} Q(t)^{(n-k)}\sigma_{2n}[k]. \tag{$\dagger$}$$

Unfortunately, careful calculations show us that the later two expansions are not proper and need much larger amount of complex calculations though we can obtain some possible symmetries in principle. Here are two special truncated symmetries obtained from the above formal series symmetry expansion (†)

$$\sigma_{0t}(t) = \begin{pmatrix} aF \\ aG \end{pmatrix}, \tag{39}$$

$$\sigma_{1t}(t) = \begin{pmatrix} d(Fqx + 4F_t) \\ d(Gqx + 4G_t) \end{pmatrix}, \tag{40}$$

where $a \equiv a(t)$ and $d \equiv d(t)$ are two arbitrary functions of time t.

We have not found any truncated symmetries from the expansion ($\star$).

3. Symmetry reductions

According to the truncated symmetries presented above namely (30)–(34), (39) and (40), a more general Lie point symmetry of the bilinear (2 + 1)-dimensional ShG system (5) and (6) can be written as follows:

$$\sigma = \begin{pmatrix} \sigma_F \\ \sigma_G \end{pmatrix}, \tag{41}$$

where σ_F and σ_G are

$$\sigma_F = \left[\frac{1}{24}l_{yyy}x^4 + \frac{1}{6}k_{yy}x^3 + \frac{1}{2}(j_y - l_{yy})x^2 + \left(l_y\int q\,\mathrm{d}t - k_y + qd + g\right)x - j + f + a + k\int q\,\mathrm{d}t\right]F + 8lF_y + 4(k + l_y x)F_x + 4dF_t, \quad (42)$$

and

$$\sigma_G = \left[\frac{1}{24}l_{yyy}x^4 + \frac{1}{6}k_{yy}x^3 + \frac{1}{2}(l_{yy} + j_y)x^2 + \left(l_y\int q\,\mathrm{d}t + k_y + qd + g\right)x + j + f + a + k\int q\,\mathrm{d}t\right]G + 8lG_y + 4(k + l_y x)G_x + 4dG_t. \quad (43)$$

Though the detailed calculations to get the symmetry reductions related to the vanishing σ_F and σ_G are complicated and tedious, the procedure to get the symmetry reduction solutions is regular. So, here we just write the final results by omitting all the complicated calculations:

Case 1: $l \neq 0, d \neq 0, k \neq 0$

In this case, the reduction result of equation (5) reads

$$\left(D_\xi^2 - D_\eta\right)F_1 \cdot G_1 = 0 \quad (44)$$

while the reduction result of equation (6) is

$$D_\eta\left(D_\xi^2 - D_\eta\right)F_1 \cdot G_1 = 0, \quad (45)$$

where F_1 and G_1 are related to the original fields F and G by

$$F = F_1(\xi, \eta)\,\mathrm{e}^{p_1(x,y,t)}, \qquad G = G_1(\xi, \eta)\,\mathrm{e}^{p_2(x,y,t)} \quad (46)$$

in which

$$p_1(x, y, t) = 8\left[\frac{x^4 l_y^2}{48l^2} + \left(\frac{xk}{6l} + 1\right)\frac{l_y x^2}{2l} + \frac{kx}{l} - \frac{x^4 l_{yy}}{l} + \frac{x^2k^2}{8l^2} - \frac{x^2 j}{2l} - \frac{x^3 k_y}{l} - 2\int\frac{a}{d}\,\mathrm{d}t - 2x\int q\,\mathrm{d}t + \int\left(\frac{j}{l} - \frac{k^2}{2l^2} - \frac{f}{l}\right)\mathrm{d}y - \frac{x}{\sqrt{l}}\int\left(\frac{g}{\sqrt{l}} + \frac{1}{8}\frac{k^3}{l^{\frac{5}{2}}}\right)\mathrm{d}y + \frac{1}{2}\int\frac{k}{l^{\frac{3}{2}}}\int\left(\frac{g}{\sqrt{l}} + \frac{1}{8}\frac{k^3}{l^{\frac{5}{2}}} - \frac{1}{2}\frac{kj}{l^{\frac{3}{2}}}\right)\mathrm{d}y\,\mathrm{d}y\right],$$

$$p_2(x, y, t) = 8\left[\frac{x^4 l_y^2}{48l^2} + \left(\frac{xk}{6l} + 1\right)\frac{l_y x^2}{2l} - \frac{kx}{l} - \frac{x^4 l_{yy}}{l} - \frac{x^2 j}{2l} + \frac{x^2k^2}{8l^2} - \frac{x^3 k_y}{l} - 2\int\frac{a}{d}\,\mathrm{d}t - 2x\int q\,\mathrm{d}t + \int\left(\frac{k^2}{2l^2} - \frac{j}{l} - \frac{f}{l}\right)\mathrm{d}y - \frac{x}{\sqrt{l}}\int\left(\frac{g}{\sqrt{l}} + \frac{1}{8}\frac{k^3}{l^{\frac{5}{2}}}\right)\mathrm{d}y + \frac{1}{2}\int\frac{k}{l^{\frac{3}{2}}}\int\left(\frac{g}{\sqrt{l}} + \frac{1}{8}\frac{k^3}{l^{\frac{5}{2}}} - \frac{1}{2}\frac{kj}{l^{\frac{3}{2}}}\right)\mathrm{d}y\,\mathrm{d}y\right],$$

and

$$\xi = \frac{x}{\sqrt{l}} - \frac{1}{2}\int\frac{k}{l^{\frac{3}{2}}}\,\mathrm{d}y, \qquad \eta = -\int\frac{1}{l}\,\mathrm{d}y + 2\int\frac{1}{d}\,\mathrm{d}t. \quad (47)$$

Case 2: $l = 0, d \neq 0, k \neq 0$

In this special situation, the general symmetry expressions (42) and (43) are changed to

$$\sigma'_F = \left[\frac{1}{6}k_{yy}x^3 + \frac{1}{2}j_y x^2 + (dq + g - k_y)x + a + k\int q\,\mathrm{d}t - j + f\right]F + 4kF_x + 4dF_t \tag{48}$$

and

$$\sigma'_G = \left[\frac{1}{6}k_{yy}x^3 + \frac{1}{2}j_y x^2 + (dq + g + k_y)x + a + k\int q\,\mathrm{d}t + j + f\right]G + 4kG_x + dG_t, \tag{49}$$

while the corresponding symmetry reduction results are

$$\left(D_\eta^2 + D_\xi\right)F_1 \cdot G_1 = 0 \tag{50}$$

and

$$\begin{aligned}\big[-2kD_\eta\big(D_\xi + D_\eta^2\big) + (k_{\xi\xi}\eta^2 + 2g - 2k_\xi + 2j_\xi\eta)D_\eta + (k_\xi\eta + j)D_\eta^2\big]F_1 \cdot G_1 \\ + \big[(k_\xi\eta + j)D_\xi - \big(f_\xi + g_\xi\eta + \tfrac{1}{6}k_{\xi\xi\xi}\eta^3 + \tfrac{1}{2}j_{\xi\xi}\eta^2\big)\big]F_1 \cdot G_1 = 0,\end{aligned} \tag{51}$$

and the original fields F and G are given by

$$F = F_1(\xi, \eta)\,\mathrm{e}^{p_3(x,y,t)}, \qquad G = G_1(\xi, \eta)\,\mathrm{e}^{p_4(x,y,t)}$$

in which

$$\begin{aligned}p_3(x, y, t) = {} & -4\int \frac{a}{d}\,\mathrm{d}t - \frac{1}{4}x\int q\,\mathrm{d}t + \frac{8}{3}k_y y k^3\left(\int \frac{4}{d}\,\mathrm{d}t\right)^4 \\ & - \frac{8}{3}(j_y + k_{yy}x)\left(\int \frac{4}{d}\,\mathrm{d}t\right)^3 + k(2j_y x + k_{yy}x^2 + 2g - 2k_y)\left(\int \frac{4}{d}\,\mathrm{d}t\right)^2 \\ & + 4\left(-\frac{1}{6}k_{yy}x^3 - gx + k_y x - \frac{1}{2}j_y x^2 - f + j\right)\int \frac{1}{d}\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}p_4(x, y, t) = {} & -4\int \frac{a}{d}\,\mathrm{d}t - \frac{1}{4}x\int q\,\mathrm{d}t + \frac{8}{3}k_y y k^3\left(\int \frac{4}{d}\,\mathrm{d}t\right)^4 - \frac{8}{3}(j_y + k_{yy}x)\left(\int \frac{4}{d}\,\mathrm{d}t\right)^3 \\ & + k(2j_y x + k_{yy}x^2 + 2g + 2k_y)\left(\int \frac{4}{d}\,\mathrm{d}t\right)^2 + 4\left(-\frac{1}{6}k_{yy}x^3 - gx - k_y x\right. \\ & \left. - \frac{1}{2}j_y x^2 - f - j\right)\int \frac{1}{d}\,\mathrm{d}t,\end{aligned}$$

while

$$\xi = y, \qquad \eta = x - k\int \frac{1}{d}\,\mathrm{d}t.$$

Remark. It is quite interesting that if we take a special selection

$$k = \text{constant}, \qquad f = \text{constant}, \qquad g = j = 0,$$

then the reduced bilinear equation system (50) and (51) is just the known one for the classical Boussinesq equation [20]. In other words the equation system (1) and (2) is not only a (2 + 1)-dimensional sinh-Gordon extension but also equivalent to a novel (2 + 1)-dimensional classical Boussinesq extension.

Case 3: $l = 0, d = 0, k \neq 0$

For the third case, σ_F and σ_G are

$$\sigma_F = \left(\frac{1}{6}k_{yy}x^3 + \frac{1}{2}j_y x^2 + (g - k_y)x + f + a + k\int q\,\mathrm{d}t - j\right)F + 4kF_x, \quad (52)$$

$$\sigma_G = \left(\frac{1}{6}k_{yy}x^3 + \frac{1}{2}j_y x^2 + (k_y + g)x + f + a + k\int q\,\mathrm{d}t + j\right)F + 4kG_x \quad (53)$$

and its corresponding symmetry reduction has the form

$$[4k^2 D_y + (j - 2gk)]F_1 \cdot G_1 = 0 \quad (54)$$

and

$$[(2gk - j^2)D_t + 4k^2 D_t D_y - 2a_t j]F_1 \cdot G_1 = 0, \quad (55)$$

and F_1 and G_1 are related to F and G by

$$F = F_1(y,t)\,\mathrm{e}^{p_5(x,y,t)}, \qquad G = G_1(y,t)\,\mathrm{e}^{p_6(x,y,t)},$$

where

$$p_5(x,y,t) = -\frac{x^4 k_{yy}}{96k} - \frac{x^3 j_y}{24k} - \left(\frac{k_y}{8k} + \frac{g}{8k}\right)x^2 - \left(\frac{f}{4k} + \frac{a(t)}{4k} + \frac{1}{4}\int q\,\mathrm{d}t + \frac{j}{4k}\right)x,$$

$$p_6(x,y,t) = -\frac{x^4 k_{yy}}{96k} - \frac{x^3 j_y}{24k} + \left(\frac{k_y}{8k} - \frac{g}{8k}\right)x^2 - \left(\frac{f}{4k} + \frac{a(t)}{4k} + \frac{1}{4}\int q\,\mathrm{d}t - \frac{j}{4k}\right)x.$$

4. Summary

All in all, we have found the generalized symmetries of the bilinear ShG equations (10) and (11) namely equation (41) in combination with equations (42) and (43). The bilinear ShG system (and then the usual ShG system (3) and (4) or equivalently (1) and (2)) possesses infinitely many generalized symmetries. That means we have proved that the first one of the NKP hierarchy is a new (2 + 1)-dimensional integrable extension of the ShG equation under the symmetry meaning while its Lax integrability and its integrability under possessing N soliton meaning have been known in the literature [17, 18]. The Lie point symmetries constitute a Kac–Moody–Virasoro-type Lie symmetry algebra and the Lie point symmetries are used to find the corresponding symmetry reductions. As a by product, from the special reduction (50) and (51), we know that the bilinear form (10) and (11) of the first member of the negative Kadomtsev–Petviashvili hierarchy is not only a (2 + 1)-dimensional sinh-Gordon extension but also a novel (2 + 1)-dimensional classical Boussinesq extension.

The study of the integrable systems of the negative hierarchies is very difficult and there is little progress in this direction especially in high dimensions. The study on the NKP and the ShG system (1) and (2) in [17] and this paper are only two known ones. There are many important but difficult problems on high dimensional integrable negative hierarchies that have not been solved yet. In (1 + 1)-dimensional case, the integrable negative hierarchies are linked to many interesting and important problems for the usual positive integrable hierarchies. For instance, the negative hierarchies connect the negative symmetries, the inverse recursion operators, Lax pairs, gauge transformations, Möbius transformations, nonlinearizations, consistent source and pfaffianizations. However, the possible similar problems in high dimensions are almost blank. Thus, more about the high dimensional ShG system and integrable models related to other high dimensional integrable negative hierarchies are worthy of studying further.

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References

[1] Lou S Y 1993 *J. Phys. A: Math. Gen.* **26** 4387
Lou S Y 1993 *Phys. Rev. Lett.* **71** 4099
Lou S Y 1994 *J. Phys. A: Math. Gen.* **27** 3235
Lou S Y and Lin J 1994 *Phys. Lett.* A **185** 29
[2] Das A 1989 *Integrable Models* (*Lecture Notes in Physics* vol 30) (Singapore: World Scientific)
[3] Barone A, Esposito F, Magee C J and Scott A C 1971 *Riv. Nuovo Cimento* **1** 227
[4] Bishop A R and Schneider T (ed) 1981 *Solitons and Condensed Matter Physics* (Berlin: Springer)
[5] Davydov A S 1985 *Solitons in Molecular Systems Dordrecht* (Netherlands: Reidel)
[6] Grauel A 1985 *Physica* A **12** 557
[7] Gibbon J D, James I N and Moroz I M 1979 *Phys. Scr.* **20** 402
[8] Perring K K and Skyrme T H 1962 *Nucl. Phys.* **31** 550
[9] Larsen A L and Sánchez N 1996 *Phys. Rev.* D **54** 2801
[10] Zamolodchikov Al 2000 On the thermodynamics Bethe ansatz equation in sinh-Gordon model *Preprint* hep-th/0005181
[11] Timonen J, Stirland M, Pilling D J, Cheng Y and Bullough R K 1986 *Phys. Rev. Lett.* **56** 2233
[12] Fuchssteiner B 1983 *Prog. Theor. Phys.* **70** 1508
Fuchssteiner B 1979 *Nonlinear Anal. Theory Methods Appl.* **3** 849
Fokas A S and Fuchssteiner B 1981 *Phys. Lett.* A **86** 341
[13] Santini P M and Fokas A S 1988 *Commun. Math. Phys.* **115** 375
Fokas A S and Santini P M 1988 *Commun. Math. Phys.* **116** 449
Fokas A S and Santini P M 1988 *J. Math. Phys.* **29** 1606
[14] Goddard P and Olive D 1989 *Kac–Moody and Virasoro Algebras (Advanced Series in Mathematical Physics* vol 3) (Singapore: World Scientific)
[15] Konopelchenko B G and Rogers C 1991 *Physica* D **18** 267
[16] Lou S Y 1993 *J. Phys. A: Math. Gen.* **26** 789
[17] Lou S Y 1994 *Phys. Lett.* A **187** 239
Lou S Y 1998 *Phys. Scr.* **57** 481
[18] Gengenhasi, Hu X B and Wang H Y 2007 *Phys. Lett.* A **360** 439
[19] Gu C H, Guo B L, Li Y S, Cao C W, Tian C, Tu G Z, Hu H S, Guo B Y and Ge M L 1990 *Soliton Theory and its Applications* (Hangzhou: Zhejiang Publishing House of Science and Technology) p 216
[20] Dmitry P, Johan S, Franklin L and Ignace L 1997 *J. Phys. A: Math. Gen.* **30** 8705